Synchronization threshold of a coupled time-delay system

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The synchronization threshold in the general form of a one-way coupled time-delayed system is discussed. Based on the Krasovskii-Lyapunov theory, the deduction process and the application range of the synchronization threshold are given. In addition, a misuse of the synchronization threshold is presented and is illustrated by an example.

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I. INTRODUCTION

The synchronization of chaotic systems has been considered a promising research area [1,2]. The reproducibility of chaotic trajectories through chaotic motion synchronization, together with the unpredictability and randomlike appearance of chaotic trajectories, has been proposed for secure communication applications. In particular, the positive Lyapunov exponents in a time-delayed chaotic system are unlimited. Research on the synchronization of chaotic systems with time delay has attracted more attention recently [3,4].

The second type of Lyapunov method is valid for discussing the stability of a coupled synchronization system. In this Brief Report, for a general form of coupled chaotic system, we discuss the construction of the Lyapunov function and the stability threshold of synchronization.

II. SYNCHRONIZATION THRESHOLD

A general form of the identical, one-way coupled scalar time-delayed system was considered as the following [5]:

$$\dot{x} = F(x, x_\tau, p_0), \tag{1a}$$

$$\dot{y} = F(y, y_\tau, p_0 + K(y - x)). \tag{1b}$$

The small deviations $\Delta = y - x$ are governed by a linear delay differential equation

$$\dot{\Delta} = -r(t)\Delta + s(t)\Delta_\tau, \tag{2}$$

where $-r(t) = (\partial_x + K\partial_p)F(x, x_\tau, p_0)$ and $s(t) = \partial_{x_\tau}F(x, x_\tau, p_0)$.

For system (1), the synchronization thresholds of a coupled time-delayed chaotic system by two different analytical approaches were investigated [1]. One of them is based on the Krasovskii-Lyapunov theory, which represents an extension of the second Lyapunov method for the case of time-delayed differential equations [6].

By the Krasovskii-Lyapunov theory, a positive-defined function for system (2) was introduced,

$$V(t) = \frac{1}{2}\Delta^2 + \mu\int_{-\tau}^{0}\Delta^2(t+\theta)d\theta, \tag{3}$$

where $\mu > 0$ is an arbitrary positive parameter. The derivative of the function $V(t)$ along the trajectory of Eq. (2) is as follows:

$$\dot{V}(t) = -r(t)\Delta^2 + s(t)\Delta\Delta_\tau + \mu\Delta^2 - \mu\Delta_\tau^2. \tag{4}$$

If μ is taken as in [5], then

$$\mu = |s|/2 \tag{5}$$

to make sure $\dot{V}(t) < 0$, and the stability condition of Eq. (2) is obtained in the form

$$r(t) > |s(t)|. \tag{6}$$

Condition (6) is true for two cases: (a) when s=constant and $r(t)$ is variable; (b) when r=constant and $s(t)$ is variable. The proof presented in Ref. [5] is indeed true for case (a). Then μ can be considered as a constant that is independent of time. For case (b), the right-hand side of Eq. (4) is a negative-defined function for $s^2(t) < -4\mu(\mu - r) = -4(\mu - r/2)^2 + r^2$. From here, it follows that $r > |s(t)|$ (when $\mu = r/2$). Case (b) exactly corresponds to the example of the coupled Mackey-Glass systems presented in Ref. [5]. But for the general case, condition (5) is unsuitable and condition (6) is incorrect, because $\mu > 0$ is an arbitrary positive parameter, and once it is given, it is a positive constant scalar. Moreover, s and r are variables, and thus $\mu = |s|/2$ or $\mu = r/2$ is a function of t. It conflicts with $\mu > 0$ being a positive constant scalar. In the deduction of condition (5), if μ is considered as a function of t, the derivative of μ should be considered in the expression of $\dot{V}(t)$.

Suppose $\mu = g(t) > 0$; then,

$$V = \frac{1}{2}\Delta^2(t) + g(t)\int_{-\tau}^{0}\Delta^2(t+\xi)d\xi \tag{7}$$

and

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$$\dot{V} = \dot{\Delta}(t)\Delta(t) + \dot{g}(t)\int_{-\tau}^{0} \Delta^2(t-\xi)d\xi + g(t)[\Delta^2(t) - \Delta^2(t-\tau)]$$
$$= - r(t)\Delta^2 + s(t)\Delta\Delta_\tau + g(t)\Delta^2 - g(t)\Delta_\tau^2 + \dot{g}(t)\int_{-\tau}^{0} \Delta^2(t+\theta)d\theta.$$

If $\dot{g}(t) \leq 0$, for arbitrary t, we have

$$\dot{V}(t) \leq - r(t)\Delta^2 + s(t)\Delta\Delta_\tau + g(t)\Delta^2 - g(t)\Delta_\tau^2$$
$$= [- r(t) + s^2(t)/4g(t) + g(t)]\Delta^2 - g(t)[\Delta_\tau^2 - s(t)\Delta\Delta_\tau/g(t) + (s(t)\Delta/2g(t))^2]$$
$$= - [r(t) - s^2(t)/4g(t) - g(t)]\Delta^2 - g(t)[\Delta_\tau - s(t)\Delta/2g(t)]^2$$
$$\leq - [r(t) - s^2(t)/4g(t) - g(t)]\Delta^2.$$

We obtain the stability condition as

$$r(t) - s^2(t)/4g(t) + g(t) > 0 \tag{8}$$

i.e., $r(t) > s^2(t)/4g(t) + g(t)$.

Considering the minimum of function $s^2(t)/4g(t) + g(t)$, we take [supposing $s(t) \neq 0$]

$$g(t) = |s(t)|/2 \tag{9}$$

and

$$r(t) > s^2(t)/4g(t) + g(t) = |s(t)|. \tag{10}$$

If $s(t)=0$, then $r(t)>0$ is also the stability condition for system (2). Equation (10) is the same as (6) under the condition of the derivative of $g(t)$—i.e., $|s(t)|/2$, which is nonpositive. If the nonpositive condition of the derivative is not satisfied, the stability condition is not as in Eq. (10). For example, if we take $r(t)=\frac{1}{\Delta^2}$, $s(t)=-\frac{1}{\Delta^2+1}$, then $r(t)>|s(t)|$ holds, but system (2) is unstable as $-\Delta\frac{1}{\Delta^2}-\Delta\frac{1}{\Delta^2+1}=0$ has no solution. There is no equilibrium point in system (2); nor is it stable. For a system with $r(t)=\frac{1}{\Delta^2+0.0010}$, $s(t)=-\frac{1}{\Delta^2+0.0011}$ and the initial condition of $\Delta=-0.5(-\tau\leq t\leq 0)$, here $\tau=1$, we obtain the waveform diagram as in Fig. 1. The waveform indicates that the system is unstable.

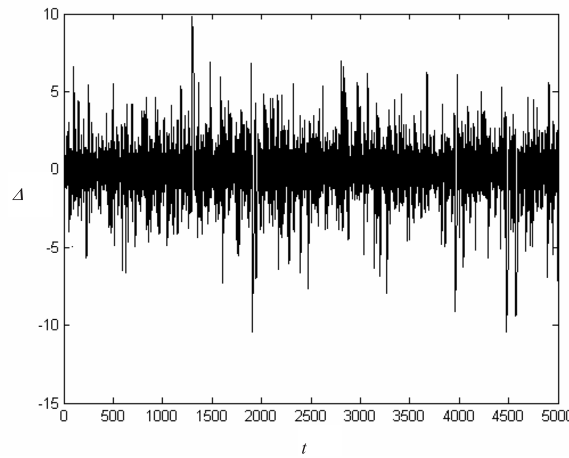

FIG. 1. Waveform of system (2) with $r(t)=\frac{1}{\Delta^2+0.0010}$, $s(t)=-\frac{1}{\Delta^2+0.0011}$.

From the above discussion, we conclude that, for the general case when both $r(t)$ and $s(t)$ are time-dependent functions, the criterion $r(t)>|s(t)|$ is not suitable.

III. CONCLUSION

In this Brief Report, we investigated the synchronization of a rather general form of the identical, one-way coupled scalar time-delayed system. The synchronization thresholds of the coupled system are obtained, and the special cases were discussed. The example presented shows that when both $r(t)$ and $s(t)$ are time-dependent functions, the criterion $r(t)>|s(t)|$ is not suitable for the general case. On the other hand, for some systems [5,7], when $|s(t)|\leq M$ holds, we could take $g(t)=M/2$ and the stability condition is $r(t)>M$.

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